Real Algebraic Geometry and Singularities

Conference in honor of
Wojciech Kucharz's 70^{th} birthday

Kraków, September 12-17, 2022

Abstracts of plenary talks

$$z = \frac{x^3}{x^2 + y^2}$$

Jagiellonian University, 2022

About the conference

Conference website:

https://realgeoms2022.c.matinf.uj.edu.pl

Organizing institution:

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Łojasiewicza 6, 30-348 Kraków,
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János Kollár (Princeton University)
Adam Parusiński (Université Côte d'Azur)
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Jagiellonian University, Faculty of Mathematics and Computer Science

Foreword

The conference *Real algebraic geometry and singularities* is organized to honor professor Wojciech Kucharz who celebrated his 70th birthday in January this year. A mathematician of renown, he does not need further presentation. We wish him many happy returns of the day and to all the participants – have a nice stay in Kraków!

The organizers

Contents

Abstracts

Homology of symmetric semi-algebraic sets

13 Sep
10:30
0004

Saugata Basu
Purdue University

Studying the homology groups of semi-algebraic subsets of $\mathbb{R}^n$ and obtaining upper bounds on the Betti numbers has been a classical topic in real algebraic geometry beginning with the work of Petrovskii and Oleinik, Thom, and Milnor. In this talk I will consider semi-algebraic subsets of $\mathbb{R}^n$ which are defined by symmetric polynomials and are thus stable under the standard action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{R}^n$. The homology groups (with rational coefficients) of such sets thus acquire extra structure as $\mathfrak{S}_n$-modules leading to possible refinements on the classical bounds. I will also mention some algorithmic consequences and connections with a homological stability conjecture.

Joint work (separately) with Daniel Perrucci and Cordian Riener.

Algebraizable classes of compact manifolds

15 Sep
10.30
0004

Olivier Benoist
École Normale Supérieure – Paris

A mod 2 cohomology class of a compact manifold M is said to be algebraizable if it is algebraic on some real algebraic model of M. I will give new examples of algebraizable and non-algebraizable classes. Our main tool are Steenrod operations.

On Hartogs' type problems in real algebraic geometry

12 Sep
10:30
0004

Jacek Bochnak
Vrije Universiteit Amsterdam

Let X be a real analytic (respectively algebraic, resp. Nash) manifold, and let $\mathcal{D}$ be a family of analytic (resp. algebraic, resp. Nash) subsets of X, which are either of dimension 1 or dimension 2.

Given a function $f\colon X \to \mathbb{R}$ such that for every set $C \in \mathcal{D}$ the restriction $f|_C$ is analytic (resp. regular, resp. Nash), we shall discuss the problem whether f itself is analytic (resp. regular, resp. Nash).

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14 Sep
11:45
0004

Birational involutions of the real projective plane that do not fix irrational curves

Ivan Cheltsov
University of Edinburgh

We present classification of birational involutions of the real projective plane up to conjugation. In contrast with an analogous classification over the complex numbers (due to E. Bertini, G. Castelnuovo, F. Enriques, L. Bayle and A. Beauville), which includes 4 different classes of involutions, there are 12 different classes over the reals. In this talk, we will explain how to classify birational involutions that do not fix (pointwise) any irrational curves. This is a joint work with Frédéric Mangolte, Egor Yasinsky and Susanna Zimmermann.

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16 Sep
10:30
0004

Representation of positive semidefinite elements as sums of squares

José F. Fernando
Universidad Complutense de Madrid

A classical problem in real geometry consists of determining if the set $\mathcal{P}(A)$ of positive semidefinite elements of A coincides with the set ΣA^2 of sums of squares of A. If A is either an excellent ring of dimension ≥ 3 or a non real-reduced local ring with non-empty real spectrum, then $\mathcal{P}(A) \neq \Sigma A^2$. Recall that the real spectrum of A is empty if and only if -1 is a sum of squares in A. If in addition $\frac{1}{2} \in A$, then each $a \in A$ satisfies

$$a = \Big(\frac{a+1}{2}\Big)^2 + (-1)\Big(\frac{a-1}{2}\Big)^2,$$

so $A = \Sigma A^2$ (this happens for instance for all rings of prime characteristic $p \geq 3$). If A is a ring of characteristic $p = 2$, sums of squares in A are squares in A (because double products are 0) and each element $a \in A$ is a square in A if and only if the Frobenius homeomorphism $\varphi\colon A \to A,\ x \mapsto x^2$ is surjective, that is, A is a semiperfect ring.

Thus, we focus on the study of the property $\mathcal{P}(A) = \Sigma A^2$ for real-reduced excellent henselian local rings $(A, \mathfrak{m})$ of dimension ≤ 2, embedding dimension ≤ 3, with non-empty real spectrum, residue field $\kappa := A/\mathfrak{m}$ and such that $\frac{1}{2} \in A$ (in particular, κ has characteristic 0). We obtain the following two main results:

Theorem 1. *If A has dimension 1 and $\theta\colon \widehat{A} \hookrightarrow \widehat{A}'$ is the normalization of the completion $\widehat{A}$ of A, then $\mathcal{P}(A) = \Sigma A^2$ if and only if one of the following situations are satisfied:*

(i) $\widehat{A}$ is an integral domain and θ is equivalent to one of the following:

*(1) $\theta\colon \widehat{A} \hookrightarrow \widehat{A}' = \kappa[[\mathtt{t}]],\ (\mathtt{x}, \mathtt{y},$
$mathttz) \mapsto (\mathtt{t}, 0, 0)$.*

(2) $\theta\colon \widehat{A} \hookrightarrow \widehat{A}' = \kappa[\sqrt{a}][[\mathtt{t}]],\ (\mathtt{x}, \mathtt{y}, \mathtt{z}) \mapsto (\mathtt{t}, \sqrt{a}\mathtt{t}, 0)$ for some $a \notin -\Sigma\kappa^2$.

(3) $\theta\colon \widehat{A} \hookrightarrow \widehat{A'} = \kappa[\theta][[\mathtt{t}]]$, $(\mathtt{x},\mathtt{y},\mathtt{z}) \mapsto (\mathtt{t},\rho\mathtt{t},\rho^2\mathtt{t})$ *for some* $\rho \in \overline{\kappa}$ *such that the extension of fields* $\kappa[\rho]|\kappa$ *has degree* 3.

(ii) $\widehat{A}$ *is not an integral domain and it is equivalent to one of the following rings:*

(1) $\kappa[[\mathtt{x},\mathtt{y}]]/(\mathtt{x}^2-\mathtt{y}^2)$.

(2) $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{x}^2-a\mathtt{y}^2,\mathtt{xz},\mathtt{yz})$ *for some* $a \notin -\Sigma\kappa^2$.

Theorem 2. *If A has dimension 2, the property $\mathcal{P}(A) = \Sigma A^2$ holds if and only if the completion $\widehat{A}$ of A is isomorphic to:*

(1) $\kappa[[\mathtt{x},\mathtt{y}]]$ *or*

(2) $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{zx},\mathtt{zy})$ *or*

(3) $\kappa[[\mathtt{x},\mathtt{y},\mathtt{z}]]/(\mathtt{z}^2-F(\mathtt{x},\mathtt{y}))$

where $F \in \kappa[[\mathtt{x},\mathtt{y}]]$ is one of the series in the following list:

(i) $a\mathtt{x}^2+b\mathtt{y}^{2k}$ *where* $a \notin -\Sigma\kappa^2$, $b \neq 0$ *and* $k \geq 1$,

(ii) $a\mathtt{x}^2+\mathtt{y}^{2k+1}$ *where* $a \notin -\Sigma\kappa^2$ *and* $k \geq 1$,

(iii) $a\mathtt{x}^2$ *where* $a \notin -\Sigma\kappa^2$,

(iv) $\mathtt{x}^2\mathtt{y}+(-1)^k a\mathtt{y}^k$ *where* $a \notin -\Sigma\kappa^2$ *and* $k \geq 3$,

(v) $\mathtt{x}^2\mathtt{y}$,

(vi) $\mathtt{x}^3+a\mathtt{x}\mathtt{y}^2+b\mathtt{y}^3$ *irreducible,*

(vii) $\mathtt{x}^3+a\mathtt{y}^4$ *where* $a \notin -\Sigma\kappa^2$,

(viii) $\mathtt{x}^3+\mathtt{x}\mathtt{y}^3$,

(ix) $\mathtt{x}^3+\mathtt{y}^5$.

Saturation, seminormalization and homeomorphisms of algebraic varieties

12 Sep 11:45 0004

Goulwen Fichou
Université Rennes I

We address the question under which conditions a bijective morphism between complex algebraic varieties is an isomorphism. Our two answers involve the seminormalization and saturation for morphisms between varieties, together with an interpretation in terms of continuous rational functions on the complex points of the variety. We propose also a version for algebraic varieties defined on an algebraically closed field of characteristics zero. (Joint work with François Bernard, Jean-Philippe Monnier and Ronan Quarez).

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16 Sep
9:00
0004

Smooth approximations in PL geometry

Riccardo Ghiloni
University of Trento

Let $Y \subset \mathbb{R}^n$ be a triangulable set and let r be either a positive integer or $r = \infty$. We say that Y is a $\mathscr{C}^r$-approximation target space, or a $\mathscr{C}^r$-`ats` for short, if it has the following universal approximation property: *for each $m \in \mathbb{N}$ and each locally compact subset X of $\mathbb{R}^m$, each continuous map $f\colon X \to Y$ can be approximated by $\mathscr{C}^r$ maps $g\colon X \to Y$ with respect to the strong Whitney $\mathscr{C}^0$ topology.*

Taking advantage of new approximation techniques we prove:

Theorem 1. *If Y is $\mathscr{C}^r$ triangulable in a weak sense, then Y is a $\mathscr{C}^r$-`ats`.*

This result applies to relevant classes of triangulable sets, namely:

- every locally compact polyhedron is a $\mathscr{C}^\infty$-`ats`,
- every set that is locally $\mathscr{C}^r$ equivalent to a polyhedron is a $\mathscr{C}^r$-`ats` (this includes $\mathscr{C}^r$ submanifolds with corners of $\mathbb{R}^n$) and
- every locally compact locally definable set of an arbitrary o-minimal structure is a $\mathscr{C}^1$-`ats` (this includes locally compact locally semialgebraic sets and locally compact subanalytic sets).

In addition, we prove:

Theorem 2. *If Y is a global analytic set, then each proper continuous map $f\colon X \to Y$ can be approximated by proper $\mathscr{C}^\infty$ maps $g\colon X \to Y$.*

Explicit examples show the sharpness of our results.

The results presented in this talk are obtained jointly with José F. Fernando and are contained in

- Smooth approximations in PL geometry. *Amer. J. Math.* 144 (2022), no. 4, 967–1007,

and they generalize our related results in

- Differentiable approximation of continuous semialgebraic maps. *Selecta Math.* (N.S.) 25 (2019), no. 3, Paper No. 46, 30 pp.

On the Fukui-Kurdyka-Paunescu Conjecture

13 Sep
11:45
0004

Zbigniew Jelonek
Polish Academy of Sciences

We prove the Fukui-Kurdyka-Paunescu Conjecture, which says that subanalytic arc-analytic bi-Lipschitz homeomorphisms preserve the multiplicities of real analytic sets. We also prove several other results on the invariance of the multiplicity (resp. degree) of real and complex analytic (resp. algebraic) sets. For instance, still in the real case, we prove a global version of the Fukui-Kurdyka-Paunescu's Conjecture.

In the complex case, one of the results that we prove is the following: If $(X, 0) \subset (\mathbb{C}^n, 0), (Y, 0) \subset (\mathbb{C}^m, 0)$ are germs of analytic sets and $h\colon (X, 0) \to (Y, 0)$ is a semi-bi-Lipschitz homeomorphism whose graph is a complex analytic set, then the germs $(X, 0)$ and $(Y, 0)$ have the same multiplicity. One of the results that we prove in the global case is the following: If $X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m$ are algebraic sets and $\phi\colon X \to Y$ is a semialgebraic semi-bi-Lipschitz homeomorphism such that the closure of its graph in $\mathbb{P}^{n+m}(\mathbb{C})$ is an orientable homological cycle, then $\deg(X) = \deg(Y)$.

Analytic functions and Nash functions along curves

12 Sep
9:00
0004

Krzysztof Kurdyka
Université Savoie Mont Blanc

I will describe some recent results obtained with Wojciech Kucharz. Let X be a real analytic manifold. A function $f\colon X \to \mathbb{R}$ is said to be curve-analytic if it is real analytic when restricted to any locally irreducible real analytic curve in X. We prove that every curve-analytic function with subanalytic graph is actually real analytic. To accomplish this task, we give a criterion for an arc-analytic function to be real analytic. A function is called arc-analytic if it is real analytic along any parametrized real analytic arc. We also obtain analogous results for Nash manifolds and Nash functions, in which case the assumption of subanalyticity is superfluous.

Birational involutions of the real projective plane that fix an irrational curve

14 Sep
10:30
0004

Frédéric Mangolte
Université Aix-Marseille

We present classification of birational involutions of the real projective plane up to conjugation. In contrast with an analogous classification over the complex numbers (due to E. Bertini, G. Castelnuovo, F. Enriques, L. Bayle and A. Beauville), which includes 4 different classes of involutions, there are 12 different classes over the reals. In this talk, we will explain how to classify birational involutions that fix (pointwise) an irrational curve. This is a joint work with Ivan Cheltsov, Egor Yasinsky and Susanna Zimmermann.

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14 Sep
9:00
0004

Sharply o-minimal structures and Wilkie Conjecture

Dmitry Novikov
Weizmann Institute of Science

This is a joint work with Gal Binyamini and Benny Zack.

The theory of o-minimal structures provides a powerful framework for the study of geometrically tame structures. In the past couple of decades a deep link connecting o-minimality to algebraic and arithmetic geometry has been developing. However, the axioms of o-minimality do not fully capture some algebro-arithmetic aspects of tameness that one may expect in structures arising from geometry. We propose a notion of sharply o-minimal structures refining the standard axioms of o-minimality and prove Wilkie Conjecture using properties of these structures.

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16 Sep
11:45
0004

Tame geometry in Hensel minimal fields

Krzysztof Jan Nowak
Jagiellonian University

In my talk, I will present tame topology in Hensel minimal structures from my recent papers. The axiomatic theory of those structures, introduced in a recent article by Cluckers-Halupczok-Rideau, seems to be a suitable non-Archimedean counterpart of o-minimality from real algebraic geometry. I will additionally require that every definable subset in the imaginary sort RV be already definable in the pure valued field language. This condition is satisfied by many of the classical tame structures on Henselian fields (including Henselian fields with analytic structure, V-minimal fields and polynomially bounded o-minimal structures with a convex subring), and ensures that the residue field is orthogonal to the value group.

The main results considered here will be, among others, the theorem on existence of the limit, curve selection, the closedness theorem, several non-Archimedean versions of the Łojasiewicz inequalities as well as an embedding theorem for regular definable spaces, the definable ultranormality and ultraparacompactness of definable Hausdorff LC-spaces and the theorems on extending continuous definable functions and on existence of definable retractions. Besides, the closedness theorem and ultraparacompactness of definable LC-spaces are essential ingredients for my definable non-Archimedean version of Bierstone–Milman's desingularization algorithm, which provides resolution of singularities and transformation to normal crossings by blowing up.

Arc-smooth functions and cuspidality of sets

15 Sep
11:45
0004

Armin Rainer
Universität Wien

On closed subsets X of $\mathbb{R}^d$ there are different notions of smooth functions, such as:

1. Arc-smooth functions, i.e., those functions f whose composition $f \circ c$ with all smooth curves c in X is smooth.
2. Functions that are locally smoothly extendable to the ambient space.
3. Functions that are smooth in the interior of X and the derivatives of all orders extend continuously to the boundary of X. This is only meaningful if X is fat (i.e., the closure of its interior).

We will show that all three notions coincide on sufficiently tame closed sets, e.g., sets with Hölder continuous boundary or fat subanalytic sets (satisfying an additional natural topological condition). This generalizes a classical theorem of Boman which states that on open sets the arc-smooth functions are just the functions that are smooth in the usual sense. On closed sets with infinitely flat cusps the notions fall apart.

Interestingly, on such tame closed sets arc-smooth functions that are also arc-analytic have holomorphic extensions to some neighborhood of X in $\mathbb{C}^d$.

These results are qualitative in nature. In a more quantitative regard, we can ask to which order of differentiation of $f \circ c$ we have to go in order to determine the derivatives of f. Typically, this involves a certain loss of regularity which is related to the cuspidality of the set X. We will make this relationship precise and show its general optimality.

Spectrahedra and their shadows

13 Sep
9:00
0004

Claus Scheiderer
Universität Konstanz

A spectrahedron is the solution set of a linear matrix inequality, or equivalently, an affine-linear section of the cone of positive semidefinite matrices of some size. A spectrahedral shadow is a linear image of a spectrahedron. The interest in spectrahedra comes originally from convex optimization and semidefinite programming. In my talk I will largely ignore this perspective, and will try to demonstrate that spectrahedra are highly interesting objects of real algebraic geometry by themselves. The talk will be a survey that addresses important results and open questions related to spectrahedra.

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15 Sep
9:00
0004

Levels of function fields of real varieties

Olivier Wittenberg
Université Sorbonne Paris Nord

Let X be a smooth real algebraic variety of dimension d. It has been known since Artin that -1 can be written as a sum of squares in the function field of X if and only if X has no real point. Under the hypothesis that X has no real point, what is then the minimum number of squares needed for this? We exhibit a link between this question and the geometry and cohomology of X, by showing that Pfister's upper bound 2^d can be improved under various sets of assumptions on X. This is joint work with Olivier Benoist.

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