Real Algebraic Geometry and Singularities

Conference in honor of
Wojciech Kucharz's 70^{th} birthday

Kraków, September 12-17, 2022

Abstracts of contributed talks

Jagiellonian University, 2022

About the conference

Conference website:

https://realgeoms2022.c.matinf.uj.edu.pl

Organizing institution:

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Łojasiewicza 6, 30-348 Kraków,
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Jagiellonian University, Faculty of Mathematics and Computer Science

Foreword

The conference *Real algebraic geometry and singularities* is organized to honor professor Wojciech Kucharz who celebrated his 70th birthday in January this year. A mathematician of renown, he does not need further presentation. We wish him many happy returns of the day and to all the participants – have a nice stay in Kraków!

The organizers

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Abstracts

Regulous functions in complex algebraic geometry

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François Bernard
Université d'Angers

The theory of regulous functions has been introduced around 2015 in the context of real algebraic geometry and it has been extensively studied, notably by W. Kucharz, since then. For a complex algebraic variety, one can also consider rational functions which extend continuously on the closed points of the variety for the euclidean topology. However, the complex case appears to be quite different from the real one. For example, there is no difference between continuous rational functions and regulous functions. Also, if the variety is smooth, those are just polynomial functions. Nevertheless, we will see that the sheaf of regulous functions over a complex algebraic variety X is not trivial and that it corresponds to the structural sheaf of a slightly less singular variety called the 'seminormalization' of X. This seminormalization can be obtained by an algebraic process and it has the property to be the biggest variety between X and its normalization such that its closed points are in bijection with those of X. Through the talk, we will provide several examples and several characterizations of complex regulous functions.

C-holomorphic functions with algebraic graphs

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Adam Białożyt
Jagiellonian University

C-holomorphic functions defined on algebraic sets and having algebraic graphs can be considered complex kin to regulous functions introduced recently in real geometry. Similarly to its real counterpart, this ring of functions admits a number of remarkable results such as the Nullstellensatz, Bezout-type inequalities or the existence of the Łojasiewicz exponent. During the talk we will introduce the notion and cast light on fundamental problems one must be aware of while investigating this family of functions, and methods employed to tackle them.

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Three Hypotheses on the Łojasiewicz Exponent

Szymon Brzostowski
University of Łódź

Let $f\colon (\mathbb{C}^n;0) \to (\mathbb{C};0)$ be a holomorphic function possessing an isolated critical point at the origin. In this talk, we will discuss three conjectures concerning the Łojasiewicz exponent $ł(f)$ of the function f . Talking the evidence gathered in favour of the conjectures, we will in particular focus on recent joint results with T. Krasiński and G. Oleksik concerning Kushnirenko non- degenerate singularities, as well as on the effects of empirical research conducted with T. Rodak for the elements of the Arnold's 'zoo'.

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On Nash images of closed balls

Antonio Carbone
University of Trento

In June 1990, during the *Oberwolfach reelle algebraische geometrie* week, Gamboa proposed to characterise those semi-algebraic subsets of $\mathbb{R}^n$ that are polynomial images of some Euclidean space. In the last 25 years this problem has been intensively investigated, mainly by Fernando, Gamboa and Ueno, and relevant results achieved, but the general question is far to be understood and the original problem is still widely open. In the last years two strategies have been used in order to have a better understanding of the polynomial images of Euclidean spaces:

- the use of suitable compact models (closed balls, spheres etc.)
- the use of bigger class of algebraic functions (regular, rational continuous, Nash etc.).

In a joined work with Fernando, we adopted a mixed strategy investigating the Nash images of the closed ball. This work has been motivated by a recent work of Fernando and Ueno on polynomial images of the closed ball and by a work of Fernando where he achieved a full characterisation of Nash images of Euclidean spaces (solving a conjecture of Shiota). Unlike the polynomial case, by taking advantage of a less rigidity of the Nash functions, we have been able to give a complete characterisation of Nash images of the closed ball. This characterisation depends only on the dimension of the given semi-algebraic set and its connection by analytic paths. Moreover we proved that our result does not depend on the chosen compact model which is another significant difference respect to the polynomial case.

Algebraic cycles on real abelian threefolds

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Olivier de Gaay Fortman
École Normale Supérieure – Paris

The goal of this talk is to prove that modulo torsion, real abelian threefolds satisfy the real integral Hodge conjecture introduced by Benoist and Wittenberg.

To be precise, let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ be the Galois group of $\mathbb{C}$ over $\mathbb{R}$. We will show that for every abelian variety A of dimension three over $\mathbb{R}$, the homomorphism

$$\mathrm{CH}_1(A) \to \mathrm{Hdg}^4(A(\mathbb{C}), \mathbb{Z})^G$$

from the Chow group of curves on A, to the group of G-invariant integral Hodge classes of degree four in the singular cohomology ring of $A(\mathbb{C})$, is *surjective*.

By work of Krasnov, Mangolte and Van Hamel, the analogous question for divisors is true. Therefore, every G-invariant integral Hodge class on a real abelian threefold is an integral linear combination of classes attached to real algebraic subvarieties.

One may wonder if, more generally, real abelian threefolds satisfy the real integral Hodge conjecture. By the above, one has to consider torsion classes in equivariant cohomology. If time permits, we show that many of them are indeed algebraic.

On singularities of the Gauss map components of surfaces in $\mathbb{R}^4$

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Wojciech Domitrz
Warsaw University of Technology

This is a joint work with L. I. Hernández-Martínez and F. Sánchez-Bringas. We study generic singularities of components of the Gauss map of a smooth, oriented surface immersed in 4-dimensional Euclidean space. These components are smooth maps into the 2-dimensional sphere. We prove that their generic singularities are folds and cusps. We also present Gauss-Bonnet type formulas for these components.

On some regularity condition

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Beata Gryszka
Pedagogical University of Kraków

We will present a theorem, which says that if $\mathbb{K}$ is a field of characteristic zero, a function $f\colon \mathbb{K}^n \to \mathbb{K}$ has a rational representation and the restriction of f to every vector plane contained in $\mathbb{K}^n$ is regular, then f is regular at the origin. During the talk we will also show that if $\mathbb{K}$ is uncountable and the restriction of $f\colon \mathbb{K}^n \to \mathbb{K}$ to every affine plane is regular, then f is regular. In this theorem we do not have to assume that f has a rational representation.

This is a joint work with J. Gwoździewicz.

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Criteria for Algebraicity of Analytic Functions

Janusz Gwoździewicz
Pedagogical University of Kraków

We consider functions defined on an open subset of a nonsingular, either real or complex, algebraic set. We give criteria for an analytic function to be a Nash (resp. regular, resp. polynomial) function. Our criteria depend only on the behavior of such a function along irreducible nonsingular algebraic curves passing trough a given point.

This is a joint work with Jacek Bochnak and Wojciech Kucharz.

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Integrating Nash functions – an algebraic point of view

Zbigniew Hajto
Jagiellonian University

In my talk, I will present Galois theory for partial differential systems defined over formally real differential fields with a real closed field of constants and over formally p-adic differential fields with a p-adically closed field of constants. For an integrable partial differential system defined over such a field, there exists a formally real (resp. formally p-adic) Picard-Vessiot extension. Moreover, I will comment on the uniqueness result for these Picard-Vessiot extensions and on the Galois correspondence theorem in this setting. I will explain the application of this theorem to characterise formally real Liouvillian extensions of real partial differential fields with a real closed field of constants by means of split solvable linear algebraic groups. In this context, Nash functions will be characterised by finite algebraic extensions. Finally, I will discuss some possibilities for further development of this theory.

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Sums of even powers of k-regulous functions

Tomasz Kowalczyk
Jagiellonian University

Artin proved that any nonnegative polynomial p on $\mathbb{R}^n$ can be written as a sum of squares in the field of rational functions $\mathbb{R}(x_1, x_2, \ldots, x_n)$, i.e.

$$p = \sum_{i=1}^{n} \left(\frac{f_i}{g_i} \right)^2,$$

where $f_i, g_i \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. One can consider the zero set of the ideal generated by all possible denominators in all possible expressions as above – these are the so called 'bad points' of p. Delzell proved that for a nonnegative polynomial p such set has codimension at least three. The situation becomes very different if we allow p to be a k-regulous function. If $k = 0$ then there are no bad points, but if $k > 0$ then

the set of bad points can have codimension 2. Besides the above, I will also discuss the 2-Pythagoras number of the ring of 0-regulous functions on an affine 0-regulous variety $X - \mathcal{R}^0(X)$ and higher Pythagoras numbers of the ring $\mathcal{R}^k(\mathbb{R}^n)$ and show its connections with the space of homogenous polynomials of degree k.

Based on joint work with J. Banecki

Jumps of the Milnor number in deformations of singularities

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Tadeusz Krasiński
University of Łódź

The Milnor number of an isolated singularity $V(f)$ is one of the most important topological invariant of $V(f)$. In the lecture we describe the behaviour (jumps) of the Milnor number in deformations of plane curve singularities.

This is a joint work with Justyna Walewska.

Betti Numbers of Random Hypersurface Arrangements

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Abhiram Natarajan
University of Warwick

We study the expected behavior of the Betti numbers of arrangements of the zeros of random (distributed according to the Kostlan distribution) polynomials in $\mathbb{R}P^n$. Using a random spectral sequence, we prove an asymptotically exact estimate on the expected number of connected components in the complement of s such hypersurfaces in $\mathbb{R}P^n$. We also investigate the same problem in the case where the hypersurfaces are defined by random quadratic polynomials. In this case, we establish a connection between the Betti numbers of such arrangements with the expected behavior of a certain model of a randomly defined geometric graph. While our general result implies that the average zeroth Betti number of the union of random hypersurface arrangements is bounded from above by a function that grows linearly in the number of polynomials in the arrangement, using the connection with random graphs, we show an upper bound on the expected zeroth Betti number of random quadrics arrangements that is sublinear in the number of polynomials in the arrangement. This bound is a consequence of a general result on the expected number of connected components in our random graph model which could be of independent interest.

Joint work with Saugata Basu and Antonio Lerario.

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Taming topology

Artur Piękosz
Kraków University of Technology

In semialgebraic or o-minimal geometry, we are usually interested in neither arbitrary continuous functions nor families of subsets of the considered space that are topologies. That is why the idea of taming the notion of a topology has emerged. Eliminating the condition of being closed under infinite unions, we get the definition of a small space, often unnamed in the literature. Small spaces always have their spectral counterparts. We drop the condition that the whole space belongs to the considered lattice of subsets, which gives us a definition of a locally small space. That enables us to glue together infinite families of definable sets in structures with topologies. Due to a concrete isomorphism of constructs, our language of smopologies simplifies the language of certain Grothendieck sites used in semialgebraic and o-minimal homotopy theory without losing the results. We give some versions of Stone (or Priestley) duality, show that Kolmogorov locally small spaces have their almost-spectralifications (hence also their spectralifications) and get some descriptions of morphisms between such spaces. We shortly discuss Heyting small spaces and a version of Esakia duality.

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Applications of o-minimal geometry to analysis and approximation theory

Rafał Pierzchała
Jagiellonian University

I will discuss several intimately related problems in the theory of multivariate polynomial inequalities. In particular, given a map h in certain quasianalytic Denjoy–Carleman classes, I will show how to decide whether the image under h of a set satisfying Markov's inequality satisfies Markov's inequality. Our approach relies heavily on the theory of o-minimal structures.

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Nearly free and free plane curves

Piotr Pokora
Pedagogical University of Kraków

I will report on the recent progress devoted to free and nearly free plane curves with prescribed singularities. In the talk, I will focus on the situation when curves are arrangements of smooth rational curves and, in particular, I will present a complete classification of complex conic-line arrangements with nodes, tacnodes, and ordinary triple points that are free.

The talk is based on a joint project run in collaboration with Alexandru Dimca and with my students.

The topology of real algebraic sets with isolated singularities is determined by the field of rational numbers

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Enrico Savi

University of Trento

The aim of this talk is to describe the topology of real algebraic sets by means of polynomial equations whose coefficients are as simple as possible. In [4] the authors provide an effective procedure to modify the coefficients of a given system of real polynomial equations getting a new system of polynomial equations whose coefficients are real algebraic numbers, while preserving the topology of the starting common solution set. However, when trying to get equations with rational coefficients their result only applies in few cases depending on the starting polynomial equations. Therefore, to investigate the open question about the possibility of describing over $\mathbb{Q}$ the topology of real algebraic sets, we introduce the notion of $\mathbb{Q}$-determined real algebraic set. In particular, $\mathbb{Q}$-determined nonsingular real algebraic sets are in some sense the minimal class of real algebraic sets, in terms of assumptions to be required, to develop new smooth approximation techniques over $\mathbb{Q}$ with regard to [3], [5] and [1]. Then, applying the mentioned new smooth approximation techniques over $\mathbb{Q}$, we get a relative version of the classical Nash-Tognoli's theorem over $\mathbb{Q}$ (see [2]), that is:

Theorem 1. *Every compact $\mathscr{C}^\infty$ manifold $M \subset \mathbb{R}^n$ containing $\mathscr{C}^\infty$ submanifolds M_i of codimension one in general position, for $i = 1, \dots, \ell$, can be arbitrarily $\mathscr{C}^\infty$ approximated by a $\mathbb{Q}$-determined projectively $\mathbb{Q}$-closed nonsingular algebraic set $M' \subset \mathbb{R}^m$, for some $m \geq n$, containing $\mathbb{Q}$-determined nonsingular algebraic subsets M'_i of codimension one in general position, for $i = 1, \dots, \ell$, such that each M'_i approximates M_i, for every $i = 1, \dots, \ell$.*

Moreover, after interpolation techniques, resolution of singularities, applications of the above Theorem 1 and blowing down operations over $\mathbb{Q}$ we are able to get results also in non-compact and singular cases. Indeed, our main result, which I will explain in deep, is the following:

Theorem 2. *Every real algebraic set $V \subset \mathbb{R}^n$ with isolated singularities is semi-algebraically homeomorphic to a $\mathbb{Q}$-determined real algebraic set $V' \subset \mathbb{R}^m$, with m geqn. Furthermore, the homeomorphism $\phi\colon V \to V'$ we construct has the following additional properties:*

(*i*) *it preserves nonsingular points and restricts to a Nash diffeomorphism between the non-singular loci,*

(*ii*) *it extends to a semi-algebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$,*

(*iii*) *it is arbitrarily $\mathscr{C}^0$-close to the inclusion map $\iota\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^m$ on compact subsets of V and arbitrarily $\mathscr{C}^\infty$-close to ι on compact subsets of* $\mathrm{Nonsing}(V)$.

This is a joint work in progress with Riccardo Ghiloni.

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[4] A. Parusiński, G. Rond, *Algebraic varieties are homeomorphic to varieties defined over number fields*, Comment. Math. Helv. (95), n. 2, pp. 339–359, 2020.

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Cones of non-negative germs of analytic functions in dimension 2

Christoph Schulze

Technische Universität Dresden

Let P be a point on a two-dimensional real analytic manifold X. We consider the vector space of germs of analytic functions at P and the subset $\mathcal{P}^{Loc}(P)$ induced by analytic functions that are defined and non-negative on some neighbourhood of P on X. The faces of this convex cone characterize possible limit behaviour of locally non-negative analytic functions at P, as it is done by big O notation, and they carry a multiplicative structure. We present a classification of the faces of finite codimension using consistent real even clusters and we sketch why these faces admit unique factorizations into (multiplicatively) irreducible faces. This work builds upon the theory of singularities of plane curves over the complex numbers.

Effective Bertini theorem and formulas for multiplicity and the local Łojasiewicz exponent

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Stanisław Spodzieja

University of Łódź

The classical Bertini theorem on generic intersection of an algebraic set with hyperplanes states the following: *Let X be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where k is an algebraically closed field. Then there exists a hyperplane $H \subset \mathbb{P}^n_k$ not containing X and such that the scheme $H \cap X$ is regular at every point. Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$ considered as a projective space.* We show that one can effectively indicate a finite family of hyperplanes H such that at least one of them satisfies the assertion of the Bertini theorem, provided the characteristic of the field k is equal to zero. As an application of the method used in the proof we give effective formulas for the multiplicity and the Łojasiewicz exponent of polynomial mappings.

This is a joint work with Tomasz Rodak and Adam Różycki from the University of Łódź.

Compatibility between orderings and valuations in hyperfields

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Hanna Stojałowska

University of Szczecin

The concept of hyperfields was introduced by Krasner in 1956. Hyperfields are variants of fields where the operation of addition is multivalued. Since hyperfields are a generalization of fields, it is natural to ask which aspects of Artin-Schreier theory can be developed in the hyperfield case.

In the theory of ordered fields we say that a valuation v is compatible with an ordering P if its valuation ring is convex with respect to P. We will generalize this notion to the hyperfield case; however, we will observe that the general- ization is not straightforward. We will state also a version of the Baer-Krull theorem for hyperfields.

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Approximation in globally subanalytic and Denjoy-Carleman classes

Anna Valette
Jagiellonian University

Efroymson's Approximation Theorem asserts that if f is a $\mathcal{C}^0$ semialgebraic mapping on a $\mathcal{C}^\infty$ semialgebraic submanifold M of $\mathbb{R}^n$ and if $\varepsilon\colon M \to \mathbb{R}$ is a positive continuous semialgebraic function then there is a $\mathcal{C}^\infty$ semialgebraic function $g\colon M \to \mathbb{R}$ such that $|f - g| < \varepsilon$. The aim of this talk is to give some insights into the proof of generalized Efroymson's theorem to the globally subanalytic category. Our framework is however much bigger than this category since our approximation theorems hold on every polynomially bounded o-minimal structure expanding the real field that admits $\mathcal{C}^\infty$ cell decomposition. In particular, it applies to quasi-analytic Denjoy-Carleman classes. We will also establish approximation theorems for Lipschitz and $\mathcal{C}^1$ definable functions.

This is a joint work with Guillaume Valette.

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